A clean proof of the cleaning lemma

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DATE CREATED: 2025-10-10
DATE MODIFIED: 2025-10-10

Abstract

In this note, we give a structural proof of the cleaning lemma of quantum error-correcting codes.

1. Let C be an $[\![n,k]\!]$ quantum code. Let M be a subset of qubits. We say a logical Pauli operator is supported in M if on M^{c} it acts as identity. We say a logical Pauli operator is partly supported on M if on M it does not act as identity.

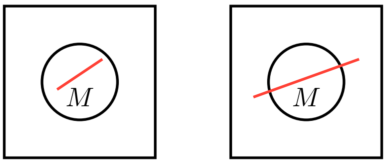

Figure 1. Left: Logical operator (red line) supported in M. Right: Logical operator partly supported on M.

2. Cleaning Lemma, cleaning version. Let C be an $[\![n,k]\!]$ quantum code. Let M be a subset of qubits in which no logical Pauli operator is supported. Then for any logical Pauli operator partly supported on M, there is an equivalent logical operator supported in M^{c}.

3. Cleaning Lemma, dimension version. Let C be an $[\![n,k]\!]$ quantum code. Let M be a subset of qubits. Let $\ell(M)$ be the number of logical operators supported in M. Then $\ell(M)+\ell(M^{\mathrm{c}})=2k$.

4. Let C be an $[\![n,k]\!]$ quantum code. The n-qubit Pauli operators are represented by vectors in a symplectic space $\mathbb{F}_2^{2n}$. Let $S\le\mathbb{F}_2^{2n}$ be its stabilizer subspace. Denote the symplectic complement of a space A be A^ω. Then the logical operators are represented in space $L:=S^\omega/S$, with $\dim L=2k$. An induced symplectic form ω_L can be defined on L by $\omega_{L(a+S,b+S)}:=\omega(a,b)$ since S is isotropic. Furthermore, ω_L is non-degenerate, which means $\forall x\in L$, $\exists y\in L$ such that $\omega_L(x,y)\ne 0$. Note that for non-degenerate symplectic form ω on V, for any subspace $W\le V$ we have $\dim W+\dim W^\omega=\dim V$.

5. Cleaning Lemma, space version. Let C be an $[\![n,k]\!]$ quantum code with stabilizer space S. Let $W\le\mathbb{F}_2^{2n}$ be any subspace. Denote $L(W):=(S^\omega\cap W)/(S^\omega\cap S)=(S^\omega\cap W+S)/S$. Then $L(W)^{\omega_L}=L(W^\omega)$.

Proof. This is almost trivial. We have a natural inclusion $\iota : L(W^\omega) \to L(W)^{\omega_L}$, $x + S \mapsto x + S$. It is well defined since if $x \in W^\omega \cap S^\omega$, then $\omega_L(x+S, y+S) = 0$ for all $y \in W \cap S^\omega$. On the other hand, this inclusion is surjective. In fact, for any $y + S \in L^{\omega_L}(W) \leq L$, by definition of L, $y \in S^\omega$; and by definition of $L(W)^{\omega_L}$, $\forall x \in W \cap S^\omega$, $\omega(x, y) = \omega_L(x+S, y+S) = 0$, leading to $y \in (W \cap S^\omega)^\omega$. Then $y \in S^\omega \cap (W \cap S^\omega)^\omega = S^\omega \cap W^\omega + S$, and $y + S \in L(W^\omega)$ and $\iota(y + S) = y + S$.

Note that in the last step we have used $(A \cap B)^\omega = A^\omega + B^\omega$. ■

6. We see immediately that Lemma 5, along with non-degeneracy of ω_L, implies Lemma 3, which implies Lemma 2.

More than Lemma 3, we have actually proved that a region M supports exactly those logical Pauli operators that commute with every logical Pauli operator supported on M^c; equivalently, $L(M) = L(M^c)^{\omega_L}$.

7. Literature. Lemma 2 can be found in [1]. Lemma 3 can be found in [2], [3] and [4]. Lemma 5 benefits from the discussion with GPT-5.

Bibliography

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